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Operators Which Do Not Have the Single Valued Extension Property

Pietro Aiena[1]

E-mail: paiena@mbox.unipa.it

and

Osmin Monsalve[2]

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Merida, Venezuela
E-mail: omonsal@ciens.ula.ve

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In this paper we shall consider the relationships between a local version of the single valued extension property of a bounded operator $T \in L(X)$ on a Banach space X and some quantities associated with T which play an important role in Fredholm theory. In particular, we shall consider some conditions for which T does not have the single valued extension property at a point $\lambda_o \in \mathbf{C}$. © 2000 Academic Press

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1. THE SVEP

The single valued extension property is a unifying theme for a wide variety of bounded linear operators. This property appeared first in Dunford [5, 6] and has received a systematic treatment in Dunford and

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Schwartz [7, 1, Part III]. In this paper we shall consider a local version of this property:

DEFINITION 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_o \in \mathbf{C}$, in short T has the SVEP at λ_o, if for every open disc $\mathbf{D}_{\lambda_o}$ centered at λ_o the only analytic function $f: \mathbf{D}_{\lambda_o} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \tag{1}$$

is the constant function $f \equiv 0$.

T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbf{C}$.

The SVEP may be characterized by means of some typical tools of the local spectral theory. To see that, let $\rho_T(x)$ denote the local resolvent set of T at the point $x \in X$, defined as the union of all open subsets $\mathcal{U}$ of $\mathbf{C}$ for which there exists an analytic function $f: \mathcal{U} \to \mathbf{C}$ which satisfies

$$(\lambda I - T)f(\lambda) = x \qquad \text{for all } \lambda \in \mathcal{U}. \tag{2}$$

The local spectrum $\sigma_T(x)$ of T at x is defined by

$$\sigma_T(x) := \mathbf{C} \setminus \rho_T(x).$$

Remark 1.2. Let $\lambda \in \rho_T(x)$ and let $\mathcal{U}$ denote an open neighborhood of λ. If $f: \mathcal{U} \to X$ satisfies the equation $(\lambda I - T)f(\mu) = x$ on $\mathcal{U}$, then $\sigma_T(f(\lambda)) = \sigma_T(x)$ for all $\lambda \in \mathcal{U}$ (see [13, Lemma 1.2.14]). Moreover, $0 \in \sigma_{\lambda I - T}(x)$ if and only if $\lambda \in \sigma_T(x)$.

THEOREM 1.3 [13, Proposition 1.2.16]. *Let $T \in L(X)$, and let X be a Banach space. Then T has SVEP if and only if every $0 \neq x \in X$ the local spectrum $\sigma_T(x)$ is non-empty.*

The SVEP has a crucial role in local spectral theory. Indeed, if this property is not enjoyed the spectrum of the operator T presents some "residual parts" which obstruct the construction of a satisfactory spectral theory (see for instance Vasilescu [21]). Consequently, it has a certain interest to find conditions for which an operator has, or does not have, the SVEP. In this paper we shall be mostly interested in finding conditions for which an operator does not have the SVEP at a point λ_o. Most of these conditions concern the ascent $p(T)$ of the operator T. There is no harm if we assume $\lambda_o = 0$; for this reason we shall limit ourselves to considering the SVEP at 0. Of course, all the statements may be formulated for an arbitrary $\lambda_o \in \mathbf{C}$ by replacing T with $\lambda_o I - T$.

The paper by Finch [8] is an important source of some results presented in this article. Our proofs will be somewhat different from those originally given by Finch and we shall emphasize, following Mbekhta [16], the role which the analytic core and the hyperrange of an operator have. Most of these proofs depend upon our main Theorem 1.9 which establishes an useful characterization of the operators which do not have the SVEP at 0.

We shall begin by introducing some T-invariant subspaces. The first one, introduced by Saphar [19], is the algebraic core of an operator.

DEFINITION 1.4. Given a vector space X and a linear operator T on X, the *algebraic core* $C(T)$ is defined to be the greatest subspace M of X for which $T(M) = M$. Equivalently (e.g., [18]),

$$C(T) = \{x \in X : \text{ there exists a sequence } (u_n) \subset X \text{ such that } x = u_o,\ Tu_{n+1} = u_n\}.$$

The second subspace that we shall consider has been studied from [16, 21] and is, in the case that T is a bounded operator on a Banach space, the analytic counterpart of $C(T)$.

DEFINITION 1.5. Let X be a Banach space and let $T \in L(X)$. The *analytical core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and $\delta > 0$ for which:

(1) $x = u_o$, and $Tu_{n+1} = u_n$ for every $n \in \mathbf{N}$.

(2) $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbf{N}$.

Remark 1.6. (a) Clearly, $K(T) \subseteq C(T)$ and it easily follows from the definition that $K(T)$ is a linear subspace of $C(T)$ and that $T(K(T)) = K(T)$.

(b) In general, neither $C(T)$ nor $K(T)$ is closed. If $C(T)$ is closed then $C(T) = K(T)$ [14].

DEFINITION 1.7. For every bounded operator $T \in L(X)$, X a Banach space, the quasi-nilpotent part of T is the set

$$H_o(T) := \left\{x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n}\right\} = 0.$$

It is easy to verify that $H_o(T)$ is a linear subspace of X. In general, $H_o(T)$ is not closed and $\ker T^m \subseteq H_o(T)$ for every $m \in \mathbf{N}$.

THEOREM 1.8. *For a bounded operator $T \in L(X)$, X a Banach space, we have*

(i) $K(T) = \{x \in X : 0 \in \rho_T(x)\}$.

(ii) $H_o(T) \subseteq \{x \in X : \sigma_T(x) \subseteq \{0\}\}$.

(iii) *If* $x \in \ker T \cap K(T)$ *then* $\sigma_T(x) = \varnothing$.

Proof. (i) and (ii) have been proved in [16]. (iii) is an obvious consequence of (i) and (ii), once it is observed that $\ker T \subseteq H_o(T)$. ∎

By Theorem 1.3 T does not have the SVEP if and only if there exists an element $0 \neq x \in X$ such that $\sigma_T(x) = \emptyset$. Next result shows a local version of this fact.

THEOREM 1.9. *Suppose that $T \in L(X)$, X a Banach space. Then T does not have the SVEP at* 0 *if and only if there exists* $0 \neq x \in \ker T$ *such that* $\sigma_T(x) = \emptyset$.

Proof. Suppose that there exists an element $0 \neq x_o \in \ker T$ such that $\sigma_T(x_o) = \emptyset$. Then, by Theorem 1.8, $x_o \in K(T)$. We can assume $\|x_o\| = 1$. By definition of $K(T)$ there exists a sequence $(u_n) \subset X$ such that

$$u_o = x_o, \qquad Tu_n = u_{n-1} \qquad \text{and} \qquad \|u_n\| \leq \delta^n,$$

for every $n = 1, 2$. Clearly, the series $\sum_{n=0}^{\infty} \lambda^n u_n$ converges for $|\lambda| < \frac{1}{\delta}$, so the function $f(\lambda) := \sum_{n=0}^{\infty} \lambda^n u_n$ is analytic on the open disc $\mathbf{D}(0, \frac{1}{\delta})$. We have

$$(\lambda I - T) \sum_{n=0}^{k} \lambda^n u_n = \lambda^{k+1} u_k$$

and $\|\lambda^{k+1} u_k\| \leq \delta^k |\lambda|^{k+1}$. Furthermore, for every $|\lambda| < \frac{1}{\delta}$ we have $\lim_{k \to \infty} \delta^k |\lambda|^{k+1} = 0$, so

$$(\lambda I - T) f(\lambda) = \lim_{k \to \infty} (\lambda I - T) \left(\sum_{n=0}^{k} \lambda^n u_n \right) = 0.$$

Since $f(0) = x_o \neq 0$, it follows that T does not have the SVEP at 0.

Conversely, suppose that for every $0 \neq x \in \ker T$ we have $\sigma_T(x) \neq \emptyset$. Consider the open disc $\mathbf{D}(0, \epsilon)$ and let $f\colon \mathbf{D}(0, \epsilon) \to X$ be an analytic function such that $(\lambda I - T) f(\lambda) = 0$ for every $\lambda \in \mathbf{D}(0, \epsilon)$. Then $f(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n$, for a suitable sequence $(u_n) \subset X$. Clearly, $Tu_o = T(f(0)) = 0$, so $u_o \in \ker T$. Moreover, from the equalities

$$\sigma_T(f(\lambda)) = \sigma_T(0) = \emptyset \qquad \text{for every } \lambda \in \mathbf{D}(0, \epsilon)$$

we obtain that $\sigma_T(f(0)) = \sigma_T(u_o) = \emptyset$, so, by assumption, we conclude that $u_o = 0$. For all $0 \neq \lambda \in \mathbf{D}(0, \epsilon)$ then we have

$$\begin{aligned} 0 = (\lambda I - T) f(\lambda) &= (\lambda I - T) \left(\sum_{n=1}^{\infty} \lambda^n u_n \right) \\ &= \lambda(\lambda I - T) \left(\sum_{n=1}^{\infty} \lambda^n u_{n+1} \right), \end{aligned}$$

and therefore

$$0 = (\lambda I - T)\left(\sum_{n=0}^{\infty} \lambda^n u_{n+1}\right) \qquad \text{for every } 0 \neq \lambda \in \mathbf{D}(0, \epsilon).$$

By continuity this is still true for every $\lambda \in \mathbf{D}(0, \epsilon)$. At this point, by using the same argument as in the first part of the proof it is possible to show that $u_1 = 0$ and, by iterating this procedure, we conclude that $u_2 = u_3 = \cdots = 0$. This shows that $f \equiv 0$ on $\mathbf{D}(0, \epsilon)$ and therefore T has the SVEP at 0. ∎

Let $\sigma_p(T)$ denote the *point spectrum* of $T \in L(X)$; i.e.,

$$\sigma_p(T) := \{\lambda \in \mathbf{C} : \lambda \text{ is an eigenvalue of } T\}.$$

The following corollary is a more detailed version of the result established in Theorem 1.3.

COROLLARY 1.10. *Let $T \in L(X)$, X a Banach space. Then T does not have SVEP if and only if there exists $\lambda_o \in \sigma_p(T)$ and a corresponding eigenvector $x_o (\neq 0)$ such that $\sigma_T(x_o) = \emptyset$. In such a case T does not have the SVEP at λ_o.*

Clearly, if T does not have the SVEP at 0 then T is not injective. The next result, which easily follows from Theorem 1.9, shows that the converse is still true if we assume that T is surjective.

COROLLARY 1.11 [8]. *Let $T \in L(X)$, X a Banach space, be surjective. Then*

$$T \text{ does not have the SVEP at } 0 \quad \Leftrightarrow \quad T \text{ is not injective}.$$

Proof. Suppose T surjective but not injective. Since T is surjective then $C(T) = X$ is closed, so, by Remark 1.6, part (b), $K(T) = X$. Hence $\{0\} \neq \ker T \subseteq K(T)$ and therefore, by Theorem 1.8, part (iii), $\sigma_T(x) = \emptyset$ for every $x \in \ker T$. ∎

Clearly, if Y is a closed subspace of X such that $T(Y) = Y$ and the restriction $T \mid Y$ does not have the SVEP at 0 then also T does not have the same property at 0.

This observation, together with Corollary 1.11, suggests how to obtain operators without SVEP: if for an operator $T \in L(X)$ there exists a closed subspace Y such that $T(Y) = Y$ and $\ker T \cap Y \neq \{0\}$ then T does not have the SVEP at 0.

2. OPERATORS WITH INFINITE ASCENT

In this section we look for a subspace Y which verifies $T(Y) = Y$ and such that $T \mid Y$ is not injective. A natural candidate is given by the following T-invariant subspace.

DEFINITION 2.1. Let T be a linear operator on a vector space X. The *hyperrange* of T is the subspace

$$T^{\infty}(X) := \bigcap_{n \in \mathbf{N}} T^{n}(X).$$

Generally, $T(T^{\infty}(X)) \subseteq T^{\infty}(X)$, so we are interested in finding conditions for which $T(T^{\infty}(X)) = T^{\infty}(X)$. Obviously, the last equality will be verified if $T^{\infty}(X) = C(T)$. A simple inductive argument shows that the inclusion $C(T) \subseteq T^{n}(X)$ holds for all $n \in \mathbf{N}$. From this it follows that $C(T) \subseteq T^{\infty}(X)$.

The next lemma shows that under certain conditions the algebraic core and the hyperrange of an operator coincide. Although it is not explicitly stated a proof of it may be extracted from the proof of [10, Hilfssatz 72.7].

LEMMA 2.2. *Let T be a linear operator on a vector space X. Suppose that there exists $m \in \mathbf{N}$ such that*

$$\ker T \cap T^{m}(X) = \ker T \cap T^{m+k}(X) \qquad \text{for all integers } k \geq 0.$$

Then $C(T) = T^{\infty}(X)$.

Now, let us introduce two classical quantities associated with an operator. To every linear operator T on a vector space X there correspond the two chains:

$$\{0\} = \ker T^{0} \subseteq \ker T \subseteq \ker T^{2} \cdots$$

and

$$X = T^{0}(X) \supseteq T(X) \supseteq T^{2}(X) \cdots .$$

The *ascent* of T is the smallest positive integer $p = p(T)$, whenever it exists, such that $\ker T^{p} = \ker T^{p+1}$. If such p does not exist we let $p = +\infty$. Analogously, the *descent* of T is defined to be the smallest integer $q = q(T)$, whenever it exists, such that $T^{q+1}(X) = T^{q}(X)$. If such q does not exist we let $q = +\infty$.

It is possible to prove that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ [10, Satz 72.3]. Note that $p(T) = 0$ means that T is injective, and $q(T) = 0$ means that T is surjective.

THEOREM 2.3. *Let T be a linear operator on a vector space X. Suppose that one of the following conditions holds*:

(i) $\dim \ker T < \infty$.

(ii) $\operatorname{codim} T(X) < \infty$.

(iii) $p(T) < \infty$.

(iv) $q(T) < \infty$.

(v) $\ker T \subseteq T^n(X)$ *for all* $n \in \mathbf{N}$.

Then $C(T) = T^\infty(X)$.

Proof. (i) Clearly, if $\ker T$ is finite dimensional there exists a positive integer m such that

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X)$$

for all integers $k \geq 0$. Hence it suffices to apply Lemma 2.2.

(ii) If $\operatorname{codim} T(X) < \infty$, for a sufficiently large m we have

$$\ker T \cap T^m(X) = \ker T \cap T^{m+k}(X)$$

for all integers $k \geq 0$ (see [10, Hilfssatz 72.7]), so we are again in the situation of Lemma 2.2.

(iii) If $p := p(T) < \infty$ then

$$\ker T \cap T^p(X) = \ker T \cap T^{p+k}(X)$$

for all integers $k \geq 0$ [10, Satz 72.1]; thus we can apply Lemma 2.2.

(iv) If $q := q(T) < \infty$ then, by definition,

$$\ker T \cap T^q(X) = \ker T \cap T^{q+k}(X)$$

for all integers $k \geq 0$, so Lemma 2.2 applies again.

(v) Trivially, if $\ker T \subseteq T^n(X)$ for all $n \in \mathbf{N}$ then

$$\ker T \cap T^n(X) = \ker T \cap T^{n+k}(X) = \ker T$$

for all integers $k \geq 0$. Hence also in this case we can apply Lemma 2.2. ∎

THEOREM 2.4. *If $T \in L(X)$, X a Banach space, then*

$$T \text{ does not have the SVEP at } 0 \quad \Rightarrow \quad p(T) = \infty. \tag{3}$$

Proof. Suppose that T does not have the SVEP at 0. Then there exists an element $0 \neq x \in \ker T$ such that $\sigma_T(x) = \emptyset$. By Theorem 1.8 $x \in K(T)$ and therefore there is a sequence $(u_n) \in X$ such that $u_o = x$ and $Tu_{n+1} =$

u_n for every $n = 0, 1, \dots$. We have

$$T^{n+1}u_n = T^n(Tu_n) = T^n u_{n-1} = \cdots = Tu_o = 0$$

and

$$T^n u_n = T^{n-1}u_{n-1} = \cdots = u_o \neq 0,$$

for every $n = 0, 1, \dots$. Hence $u_n \in \ker T^{n+1}$ whereas $u_n \in \ker T^n$. ∎

Theorem 2.4 also shows that

$$p(\lambda I - T) < \infty \quad \text{for every } \lambda \in \mathbf{C} \qquad \Rightarrow \qquad T \text{ has the SVEP.}$$

This property has been observed in [11] by using different methods. There are many examples of operators for which $p(\lambda I - T) < \infty$ for every $\lambda \in \mathbf{C}$, for instance, multipliers of commutative semi-simple Banach algebras [1] as well as generalized scalar operators and several other classes of operators studied in [11].

Theorem 2.3 establishes that under some purely algebraic conditions the hyperrange $T^\infty(X)$ verifies $T(T^\infty(X)) = T^\infty(X)$. Now we are interested in topological conditions which ensure that this space is closed. To do that, let us introduce some important classes of operators.

(a) the class of all *upper semi-Fredholm* operators

$$\Phi_+(X) := \{T \in L(X) : \dim \ker T < \infty,\ T(X) \text{ closed}\},$$

(b) the class of all *lower semi-Fredholm* operators

$$\Phi_-(X) := \{T \in L(X) : \operatorname{codim} T(X) < \infty\}.$$

The class of all *semi-Fredholm* operators is $S\Phi(X) := \Phi_+(X) \cup \Phi_-(X)$; the class of all *Fredholm* operators is $\Phi := \Phi_+(X) \cap \Phi_-(X)$. If $T \in S\Phi(X)$ the *index* of T is defined by $\operatorname{ind} T := \dim \ker T - \operatorname{codim} T(X)$. The index is an integer or $\{\pm\infty\}$.

The proof of the next lemma may be found in [10, Hilfssatz 72.7].

LEMMA 2.5. *Let T be a linear operator on the vector space X. Suppose that* $\dim \ker T < \infty$ *or* $\operatorname{codim} T(X) < \infty$. *Then*

$$p(T) < \infty \qquad \leftrightarrow \qquad \textit{the restriction } T \,|\, T^\infty(X) \textit{ is injective}.$$

THEOREM 2.6. *Let $T \in S\Phi(X)$, X a Banach space. Then the following assertions are equivalent*:

(i) *T does not have the SVEP at* 0.

(ii) $p(T) = \infty$.

(iii) 0 *is a limit point of* $\sigma_p(T)$.

Proof. The implication (i) ⇒ (ii) has been proved in Theorem 2.4.

(ii) ⇒ (i) Suppose that $p(T) = \infty$. Assume first that $T \in \Phi_+(X) \cup \Phi_-(X)$ and $Y := T^\infty(X)$. By Theorem 2.3 we have $Y = C(T)$, so $T \mid Y$ is surjective. Moreover, the powers $T^n \in \Phi_+(X) \cup \Phi_-(X)$ for every $n \in \mathbf{N}$ [3, Corollary 1.3.3] so their image spaces are closed and hence also Y is closed. By Lemma 2.5 $T \mid Y$ is not injective, so T does not have SVEP at 0.

(iii) ⇒ (i) Let $Y := T^\infty(X)$. As in the first part of the proof, Y is a Banach space and $T \mid Y$ is surjective. It is easy to verify that if $Tx = \lambda x$ for some $\lambda \neq 0$ then $x \in Y$. Hence

$$\sigma_p(T) \setminus \{0\} \subseteq \sigma_p(T \mid Y) \subseteq \sigma(T \mid Y).$$

From the assumption it follows that 0 is a limit point of $\sigma(T \mid Y)$ and hence $0 \in \sigma(T \mid Y)$, since the last set is closed. But since $T \mid Y$ is surjective then $T \mid Y$ is not injective, so T does not have the SVEP at 0.

(i) ⇒ (iii) Suppose that 0 is not a limit point of $\sigma_p(T)$ and let f: $\mathbf{D}(0, \epsilon) \to X$ be an analytic function such that $(\lambda I - T)f(\lambda) = 0$ for every $\lambda \in \mathbf{D}(0, \epsilon)$. For a sufficiently small $\epsilon > 0$ every $0 \neq \lambda$ of $\mathbf{D}(0, \epsilon)$ is not an eigenvalue of T, so $f(\lambda) = 0$ for every $0 \neq \lambda \in \mathbf{D}(0, \epsilon)$. By continuity this is still true also for $\lambda = 0$; hence T has the SVEP at 0. ∎

Theorem 2.6 improves [8, Theorems 9 and 10] which establish that T does not have the SVEP at 0 under the stronger assumption that $\sigma_p(T)$ contains a neighborhood of zero.

COROLLARY 2.7. *Suppose that $T \in S\Phi(X)$ has index* $\operatorname{ind} T > 0$. *Then T does not have the SVEP at* 0.

Proof. If $\operatorname{ind} T > 0$ then $p(T) = \infty$; see [10, Satz 72.5]. ∎

An example of an operator T which has index $\operatorname{ind} T < 0$ and does not have the SVEP at 0 may be found in [8].

THEOREM 2.8. *Suppose that $T \in S\Phi(X)$. Then*

$$T^* \textit{ does not have the SVEP at } 0 \qquad \leftrightarrow \qquad q(T) = \infty.$$

Furthermore, if T and T^* have the SVEP at 0 *then $T \in \Phi(X)$ with* $\operatorname{ind} T = 0$.

Proof. It is known that if $T \in S\Phi(X)$ then $T^* \in S\Phi(X^*)$ and $p(T^*) = q(T)$.

The last assertion is an evident consequence of the fact that if both ascent $p(T)$ and descent $q(T)$ are finite, and hence equal, then $\dim \ker T = \operatorname{codim} T(X)$ [10, Satz 72.5]. ∎

Theorem 2.8 shows also that if T and T^* have the SVEP at 0 then T is a *Riesz–Schauder* operator, i.e., a Fredholm operator having both ascent and descent finite. This has been observed, by means of a more involved argument, in [17].

If $T \in \Phi(X)$ let us consider the so-called *Fredholm set*

$$\Phi(T) := \{\lambda \in \mathbf{C} : \lambda I - T \in \Phi(X)\}.$$

It is well known that $\Phi(T)$ is an open set [10, p. 506], so it may be decomposed in connected disjoint open nonempty components.

THEOREM 2.9. *Let $T \in L(X)$, X a Banach space. Then T has the SVEP either for every point or for no point of a component Ω of $\Phi(T)$.*

Proof. The ascent $p(\lambda I - T)$ is finite either for every point or for no point of Ω [10, Satz 104.3]. Suppose that T does not have the SVEP at $\lambda_o \in \Omega$. By Theorem 2.6 $\lambda_o I - T$ has infinite ascent and hence $p(\lambda I - T) = \infty$ for every $\lambda \in \Omega$. By Theorem 2.6 it follows that T has the SVEP at no point of Ω. ∎

Theorem 2.6 has another interesting consequence. Let

$$\sigma_W(T) := \mathbf{C} \setminus \{\lambda \in \mathbf{C} : \lambda I - T \in \Phi(X),\ \operatorname{ind} T = 0\}$$

denote the *Weyl spectrum* of $T \in L(X)$ and let

$$\sigma_B(T) := \mathbf{C} \setminus \{\lambda \in \mathbf{C} : \lambda I - T \in \Phi(X),\ p(\lambda I - T) = q(\lambda I - T) < \infty\}$$

denote the *Browder spectrum* of T.

THEOREM 2.10. *Suppose that $T \in L(X)$, X a Banach space, has the SVEP. Then $\sigma_B(T) = \sigma_W(T)$.*

Proof. The inclusion $\sigma_W(T) \subseteq \sigma_B(T)$ is true for every bounded operator, since every Fredholm operator with finite ascent and descent necessarily has index 0 [10, Satz 72.6, part (a)].

Conversely, suppose that $\lambda \notin \sigma_W(T)$. Since T has the SVEP at $\lambda \in \mathbf{C}$, the ascent $p(\lambda I - T)$ is finite by Theorem 2.6; therefore also $q(\lambda I - T)$ is finite, by [10, Satz 72.6]. Hence $\lambda \notin \sigma_B(T)$. ∎

In particular, Theorem 2.10 applies to decomposable and generalized scalar operators, since these have the SVEP [13]. The equality $\sigma_B(T) = \sigma_W(T)$ has also been noted in [1, 12] for every multiplier of a semi-prime Banach algebra. Indeed, as has been observed above, these operators enjoy the SVEP.

Recall that $T \in L(X)$, X a Banach space, is said to be a *semi-regular* operator if T has closed range $T(X)$ and $\ker T \subseteq T^n(X)$ for every $n \in \mathbf{N}$. By Theorem 2.3 if T is semi-regular then $C(T) = T^\infty(X)$. Furthermore, in this case $C(T)$ is closed [18, Corollaire 2.6] and hence $C(T) = K(T)$. Examples of semi-regular operators may be found in [18].

Clearly, every surjective operator is semi-regular, so the next result generalizes Corollary 1.11.

THEOREM 2.11. *Let $T \in L(X)$, X a Banach space, be a semi-regular operator. Then*

$$T \text{ does not have the SVEP at } 0 \quad \leftrightarrow \quad T \text{ is not injective}.$$

Proof. Suppose that T is not injective. Since T is semi-regular $Y := T^{\infty}(X)$ is closed and $T^{\infty}(X) = C(T) = K(T)$; thus $T \mid Y$ is surjective. Furthermore, by assumption, $\ker T \subseteq T^n(X)$ for every $n \in \mathbf{N}$ and hence $\ker T \subseteq T^{\infty}(X)$; i.e., $T \mid Y$ is surjective but not injective, and thus T does not have the SVEP at 0. ∎

COROLLARY 2.12. *Let $T \in L(X)$, X a Banach space, be a semi-regular operator. Then*

$$T^* \text{ does not have the SVEP at } 0 \quad \leftrightarrow \quad T \text{ is not surjective}.$$

Proof. Observe first that if T is semi-regular then also T^* is semi-regular [18] and it is well known that T is surjective if and only if T^* is bounded below (i.e., T^* is injective and has closed range). Now, by semi-regularity, T^* has closed range and hence T is not surjective if and only if T^* is not injective. ∎

Let $\rho_K(T)$ denote the *Kato resolvent* of $T \in L(X)$, i.e.,

$$\rho_K(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is semi-regular}\}.$$

$\rho_K(T)$ is an open subset of $\mathbf{C}$ and hence may be decomposed in connected disjoint open nonempty components (see, for instance, [13]).

THEOREM 2.13. *Let $T \in L(X)$, X a Banach space. Then T has the SVEP either for every point or for no point of a component Ω of $\rho_K(T)$.*

Proof. Suppose that T does not have the SVEP at $\lambda_o \in \Omega$ and consider an arbitrary point λ of Ω. In order to show that T does not have the SVEP at the point λ it suffices to show, by Theorem 2.11, that $\lambda I - T$ is not injective.

By [9], the subspaces $\overline{H_o(\lambda I - T)}$ are constant on the components of $\rho_K(T)$. Moreover, the semi-regularity of $\lambda I - T$ yields the equalities

$$\overline{H_o(\lambda_o I - T)} = \overline{H_o(\lambda I - T)} = \overline{\bigcup_{n=0}^{\infty} \ker(\lambda I - T)^n},$$

(see [18]). From this it follows that $\ker(\lambda I - T) \neq \{0\}$. Indeed, if $\ker(\lambda I - T) = \{0\}$ then $\ker(\lambda I - T)^n = \{0\}$ for every $n \in \mathbf{N}$ and therefore

$$\overline{H_o(\lambda_o I - T)} = \overline{H_o(\lambda I - T)} = \{0\},$$

which is impossible, since $\{0\} \neq \ker(\lambda_o I - T) \subseteq H_o(\lambda_o I - T)$. ∎

The result of Theorem 2.11 may be extended to a class of operators which is strictly larger than the class of all semi-regular operators. Following Mbekhta [15], we shall say that $T \in L(X)$ admits a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T \mid M$ is semi-regular, and $T \mid N$ is quasi-nilpotent. Obviously, the semi-regular operators correspond, in this decomposition, to the case $M = X$ and $N = \{0\}$.

THEOREM 2.14. *Suppose that $T \in L(X)$ admits a GKD (M, N). Then the following assertions are equivalent*:

(i) *T does not have the SVEP at* 0.

(ii) *$T \mid M$ is not injective*.

(iii) *$T \mid M$ does not have the SVEP at* 0.

(iv) 0 *is a limit point of $\sigma_p(T)$*.

Proof. Note first that if T admits a GKD (M, N) then $K(T) = C(T \mid M) \subseteq M$ [16, where $C(T \mid M)$ denotes the algebraic core of $T \mid M$. Moreover,

$$K(T) \cap \ker T = K(T) \cap M \cap \ker T = K(T) \cap \ker T \mid M$$

and since $T \mid M$ is semi-regular, $\ker T \mid M \subseteq C(T \mid M) = K(T)$. From this we conclude that $\ker T \mid M = K(T) \cap \ker T$.

Now, suppose that T does not have the SVEP at 0. By Theorem 1.9, then, there exists an element $x \in \ker T$, $x \neq 0$, such that $\sigma_T(x) = \emptyset$. From part (i) of Theorem 1.8 we have $x \in K(T) = C(T \mid M)$, so that $\ker T \mid M = K(T) \cap \ker T \neq \{0\}$. This shows the implication (i) $\Rightarrow$ (ii).

To show (ii) $\Rightarrow$ (i), assume that T has a generalized Kato decomposition (M, N) with $\ker T \mid M \neq \{0\}$. By assumption $T \mid M$ is semi-regular; thus $K(T) = C(T \mid M)$ is closed. Moreover, $\ker T \mid M = K(T) \cap \ker T \neq \{0\}$. Hence, if $Y := K(T)$, $T \mid Y$ is surjective but not injective and therefore T does not have the SVEP at 0.

(iii) $\leftrightarrow$ (ii) This follows from Theorem 2.11, since by assumption $T \mid M$ is semi-regular.

(iv) $\Rightarrow$ (i) As has been observed above $K(T)$ is closed and $\ker T \mid M = \ker T \cap K(T)$. Moreover, from the equality $K(T) = C(T \mid M)$ we obtain that $T \mid K(T)$ is surjective.

Now, assume that $Tx = \lambda x$ for some $\lambda \neq 0$. Then the sequence $x_o := x$ and $x_n := x/\lambda^n$ verifies the conditions (1) and (2) of the definition of $K(T)$; thus $x \in K(T)$. This shows that $\sigma_p(T) \setminus \{0\} \subseteq \sigma_p(T \mid K(T)) \subseteq \sigma(T \mid K(T))$ and therefore $0 \in \sigma(T \mid K(T))$, since $\sigma(T \mid K(T))$ is closed and 0 is a limit point of $\sigma(T \mid K(T))$. Hence $T \mid K(T)$ is not injective and therefore $\ker T \mid M = \ker T \cap K(T) \neq \{0\}$.

The implication (i) $\Rightarrow$ (iv) has been proved in Theorem 2.6. ∎

Theorem 2.14 improves [16, Theorem 4.2] which establishes that T does not have the SVEP at 0, under the assumption that $\sigma_p(T)$ contains a neighborhood of zero.

Note that the preceding result also applies to quasi-nilpotent operators. Indeed, every quasi-nilpotent operator T admits the trivial generalized Kato decomposition $M = \{0\}$ and $N = X$ and hence, by Theorem 2.14, these operators have the SVEP at 0. From this it follows, since for these operators every $0 \neq \lambda \in \mathbf{C}$ belongs to the resolvent of T, that every quasi-nilpotent operator has the SVEP (actually we have much more: these operators are super-decomposable [13]).

Another relevant case is obtained if we assume, in the decomposition above, that $T \mid N$ is nilpotent. In this case T is said to be of *Kato type* (see [18]).

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